

On the volume of the zero cell of a class of isotropic Poisson hyperplane tessellations

Julia Hörrmann*, Daniel Hug†

Abstract

We study a parametric class of isotropic but not necessarily stationary Poisson hyperplane tessellations in n -dimensional Euclidean space. Our focus is on the volume of the zero cell, i.e. the cell containing the origin. As a main result, we obtain an explicit formula for the variance of the volume of the zero cell in arbitrary dimensions. From this formula we deduce the asymptotic behaviour of the volume of the zero cell as the dimension goes to infinity.

Key words: Poisson hyperplane tessellation; Poisson-Voronoi tessellation; Zero cell; Typical cell; Variance; High dimensions

2010 Mathematics Subject Classification: Primary: 60D05; Secondary: 52A22

1 Introduction

The majority of contributions to random tessellations is devoted to investigations in low and fixed dimensions. In particular, there exist only a few results on random tessellations in high dimensions, that is, with focus on asymptotic aspects as the dimension goes to infinity. Recently, the typical cell of a stationary Poisson-Voronoi tessellation in high dimensions has been studied in [1], [26] and [37]. Alishahi and Sharifitabar [1] investigate the asymptotic behaviour of the volume and the shape of the typical cell of a stationary Poisson-Voronoi tessellation as the dimension n of the space goes to infinity. In particular, they showed that the variance of the volume of the typical cell converges to zero exponentially fast as $n \rightarrow \infty$ whereas it is well known that the expected volume is independent of the dimension. In the course of their investigation, they made use of an explicit formula for the variance of the volume of the typical cell in arbitrary dimensions. The asymptotic behaviour of the volume of the typical cell was studied earlier in the more general context of the nearest neighbour analysis by Newman et al. in [28] and [29]. They already showed convergence in distribution, though not via the convergence of the variance.

In this work, we consider a parametric class of Poisson hyperplane tessellations and focus on the volume of the cell containing the origin (the zero cell). It is then natural to explore whether an asymptotic behaviour similar to that of the typical cell of a stationary Poisson-Voronoi tessellation is exhibited by the zero cell in the present more general class of random tessellations. An interesting family of not necessarily stationary or isotropic Poisson hyperplane tessellations is introduced in connection with the investigation of Kendall's conjecture in [10]. In the isotropic case, these Poisson hyperplane tessellations are completely determined by two parameters, the intensity $\gamma \in (0, \infty)$ and the distance exponent $r \in (0, \infty)$. For a special

*Department of Mathematics, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany, julia.hoerrmann@kit.edu

†Department of Mathematics, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany, daniel.hug@kit.edu

choice of the distance exponent and the intensity, the zero cell is equal in distribution to the typical cell of a stationary Poisson-Voronoi tessellation (cf. [10], [4, Sections 5.2 and 5.3.2]). Therefore this family of isotropic Poisson hyperplane tessellations provides a general framework for investigating tessellations in high dimensions. Trying to extend the approach of Alishahi and Sharifitabar [1] to the wider context of the zero cell of this family of Poisson hyperplane tessellations, we came across the need for a formula for the variance of the volume of the zero cell. Finding a manageable expression turned out to be a rather complex issue. In fact, the formulas presented here mark the starting point of a more detailed study of the asymptotic behaviour of characteristics of the zero cell as the dimension goes to infinity. Results concerning lower dimensional sections of the zero cell and other shape characteristics as well as a connection to the hyperplane conjecture will be considered separately.

In the following, we give a more detailed overview of our results. A precise description of the particular (parametric) model of a Poisson hyperplane tessellation used here is given in Section 2. For this model we then derive, in Section 3, an explicit expression for the expectation and estimates of the moments of the volume of the zero cell in Proposition 1. An explicit expression for the second moment and the variance of the volume of the zero cell is provided in Theorem 1. These results follow from a sequence of lemmas which make use of integral geometric transformations and the symmetries of the geometric situation. In Theorem 2 we deduce estimates for the variance of the volume of the zero cell which involve auxiliary quantities $D(n, r)$ and $E(n, r)$. These quantities have to be evaluated and estimated for a given distance exponent r and an intensity γ . In Corollary 1 we consider the zero cell of a Poisson hyperplane tessellation with constant distance exponent r . The choice $r = 1$ corresponds to a stationary Poisson hyperplane process. For constant intensity we then prove that all moments as well as the variance of the volume of the zero cell converge to infinity as the dimension n goes to infinity.

In order to fix the expected volume of the zero cell, independent of the dimension, the intensity of the underlying Poisson hyperplane process can be chosen appropriately as a function of the dimension n . However, it follows from our estimates that as long as the distance exponent r is fixed, the variance of the volume of the zero cell is still divergent as n goes to infinity. The investigation in Section 4 thus suggests that in order to ensure that the variance converges to zero, the distance exponent r has to be adjusted to the dimension n . In Corollary 2 we summarize the case where the distance exponent r is proportional to the dimension n , i.e. $r = an$ with a fixed factor $a \in (0, \infty)$. For constant intensity we show that the expectation and the moments of the volume of the zero cell now all converge to zero as the dimension n goes to infinity.

In Theorem 3 we finally consider the situation where the distance exponent r is proportional to the dimension, i.e. $r = an$ with some fixed $a > 0$, and the intensity $\hat{\gamma}(a, n)$ is chosen as a function of the dimension n and the factor a in such a way that the expected volume of the zero cell is equal to a positive constant. In this case we prove that the variance of the volume of the zero cell converges to zero at an exponential speed as $n \rightarrow \infty$. In particular, the volume of the zero cell converges in distribution. In the special case $r = n$ (i.e. $a = 1$), we recover results for the typical cell of a Poisson-Voronoi tessellation. Of course, these findings are consistent with the results obtained in [1], though the estimates for the variance found by Alishahi and Sharifitabar are sharp. The present more general approach applies to a larger class of tessellations and admits various other extensions and variations that will be discussed in detail in subsequent work.

2 Preliminaries

In the following, we mainly use the notation and terminology of the monograph [34]. We work in an n -dimensional real Euclidean vector space \mathbb{R}^n , $n \geq 2$, with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. The unit ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ centred at the origin o is denoted by B^n , its boundary is the unit sphere S^{n-1} . For $k \in \{0, \dots, n\}$, the Grassmannian of k -dimensional linear subspaces of \mathbb{R}^n is denoted by $G(n, k)$, and the affine Grassmannian of k -dimensional affine subspaces (k -flats) by $A(n, k)$; both are equipped with their standard topologies. For $u \in S^{n-1}$ and $t \in [0, \infty)$, we write

$$H(u, t) := \{x \in \mathbb{R}^n : \langle x, u \rangle = t\}, \quad H^-(u, t) := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq t\}.$$

Lebesgue measure on \mathbb{R}^n is denoted by λ . For $E \in G(n, k)$, Lebesgue measure on E is denoted by λ_E . Besides we define $S_E^{k-1} := E \cap S^{n-1}$ and $H_E(u, t) := E \cap H(u, t)$, for $u \in S_E^{k-1}$ and $k \in \{1, \dots, n\}$. The s -dimensional Hausdorff measure is denoted by \mathcal{H}^s , where $s \geq 0$. For $s = n$ we sometimes refer to it as n -dimensional volume V_n . A frequently occurring constant is the volume of the unit ball,

$$\kappa_n := \lambda_n(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}.$$

The surface area of the unit sphere S^{n-1} is given by

$$\omega_n := \mathcal{H}^{n-1}(S^{n-1}) = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

For $m \in \mathbb{N}$ and $x_1, \dots, x_m \in \mathbb{R}^n$, we denote by $[x_1, \dots, x_m]$ the convex hull and by $\text{span}\{x_1, \dots, x_m\}$ the linear hull of x_1, \dots, x_m .

The family of nonempty, compact, convex subsets of \mathbb{R}^n is denoted by \mathcal{K}^n . For a topological space (T, \mathcal{T}) , a measure is always defined on the σ -algebra $\mathcal{B}(T)$ of Borel sets of T , i.e. the smallest σ -algebra containing the open sets \mathcal{T} . We write $\mu^k := \mu \otimes \dots \otimes \mu$, with k factors μ , for the k -fold product of a measure μ . By SO_n we denote the group of proper rotations on \mathbb{R}^n , and ν_n is the unique Haar probability measure on SO_n .

The following setting has previously been considered in a more general, not necessarily isotropic framework, in the context of Kendall's problem [10] (see also [6], [4]). Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote the underlying probability space. Further, let X be a Poisson hyperplane process in \mathbb{R}^n , i.e. a Poisson point process in the space $A(n, n-1)$. Subsequently, we identify a simple counting measure with its support, so that for a Borel set $A \subset A(n, n-1)$ both notations $X(A)$ and $\text{card}(X \cap A)$ denote the number of elements of X in A . We assume that the intensity measure $\Theta(\cdot) = \mathbb{E}X(\cdot)$ of X is of the form

$$\Theta(\cdot) = \frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \in \cdot\} t^{r-1} dt \mathcal{H}^{n-1}(du) \quad (1)$$

with $\gamma > 0$ and $r \in (0, \infty)$. We refer to γ as the *intensity* and to r as the *distance exponent* of the hyperplane process X . Clearly, Θ is rotation invariant for all $r > 0$, but translation invariant only for $r = 1$. Therefore, since X is a Poisson process, X is always isotropic but stationary only for $r = 1$.

The random polytope

$$Z_0 := \bigcap_{H \in X} H^-,$$

is the *zero cell* of the hyperplane process X , where H^- denotes the (almost surely uniquely determined) closed half-space bounded by H which contains the origin. Clearly, Z_0 depends on γ and r ; although this dependence is not made explicit by our notation.

For the distance exponent $r = n$ the zero cell Z_0 is equal in distribution to the typical cell of a stationary Poisson-Voronoi tessellation (see [10]). More detailed information on the topic of random tessellations is provided, e.g., in [34], [30] and [36]. Poisson-Voronoi tessellations have been studied extensively in the literature; see, for instance, [7], [13], [20], [21], [22], [23], [24], [25], [27], [30] and [36]. Stationary Poisson hyperplane tessellations have been considered, e.g., in [14], [15], [16], [17], [18], [19], [35], [9], [11], [32], [12]. Recently, also non-stationary Poisson hyperplane tessellations have attracted some attention (cf. [32], [10]). A review of recent results on random polytopes is given in [3], [31], [33], [34, Chapter 8].

3 A general formula for the variance

Let X be a Poisson hyperplane process in \mathbb{R}^n with an intensity measure of the form (1). We assume that $\gamma > 0$ and $r \in (0, \infty)$. By Fubini's theorem and basic properties of a Poisson process, we have

$$\begin{aligned} \mathbb{E}[V_n(Z_0)^k] &= \mathbb{E}\left[\int_{\mathbb{R}^n} \mathbf{1}_{Z_0}(x_1) dx_1 \cdots \int_{\mathbb{R}^n} \mathbf{1}_{Z_0}(x_k) dx_k\right] \\ &= \int_{(\mathbb{R}^n)^k} \mathbb{P}(x_1, \dots, x_k \in Z_0) dx_1 \dots dx_k \\ &= \int_{(\mathbb{R}^n)^k} \exp\left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, x_1, \dots, x_k] \neq \emptyset\} \right. \\ &\quad \left. \times t^{r-1} dt \mathcal{H}^{n-1}(du)\right] dx_1 \dots dx_k. \end{aligned} \quad (2)$$

From (2) we now deduce a lower and an upper estimate of the moments of $V_n(Z_0)$. For $e \in S^{n-1}$, we define

$$c(n, r) := \int_{S^{n-1}} \langle e, u \rangle_+^r \mathcal{H}^{n-1}(du) = \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+n}{2})},$$

which is indeed independent of the choice of the unit vector e . The explicit value is determined by a suitable decomposition of spherical Lebesgue measure.

The following result provides estimates from above and below for the moments of the volume of the zero cell. Note that the ratio of the upper and the lower bound is given by the ratio of the corresponding values of the Gamma functions in these estimates.

Proposition 1. *For $k \in \mathbb{N}$, we have*

$$\Gamma\left(\frac{n}{r} + 1\right)^k \kappa_n^k \left(\frac{n\kappa_n r}{2\gamma c(n, r)}\right)^{\frac{kn}{r}} \leq \mathbb{E}[V_n(Z_0)^k] \leq \Gamma\left(\frac{kn}{r} + 1\right) \kappa_n^k \left(\frac{n\kappa_n r}{2\gamma c(n, r)}\right)^{\frac{kn}{r}}.$$

In particular, for $k = 1$ we get

$$\mathbb{E}[V_n(Z_0)] = \Gamma\left(\frac{n}{r} + 1\right) \kappa_n \left(\frac{n\kappa_n r}{2\gamma c(n, r)}\right)^{\frac{n}{r}}.$$

Proof. Starting with (2), introducing polar coordinates and by symmetry, we get

$$\begin{aligned}
& \mathbb{E}[V_n(Z_0)^k] \\
&= k \int_0^\infty \int_0^{s_1} \dots \int_0^{s_1} \int_{(S^{n-1})^k} \exp \left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, s_1 v_1, \dots, s_k v_k] \neq \emptyset\} \right. \\
&\quad \left. \times t^{r-1} dt \mathcal{H}^{n-1}(du) \right] s_1^{n-1} \dots s_k^{n-1} \mathcal{H}^{n-1}(dv_k) \dots \mathcal{H}^{n-1}(dv_1) ds_k \dots ds_1 \\
&\leq k \int_0^\infty \int_0^{s_1} \dots \int_0^{s_1} \int_{(S^{n-1})^k} \exp \left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, s_1 v_1] \neq \emptyset\} \right. \\
&\quad \left. \times t^{r-1} dt \mathcal{H}^{n-1}(du) \right] s_1^{n-1} \dots s_k^{n-1} \mathcal{H}^{n-1}(dv_k) \dots \mathcal{H}^{n-1}(dv_1) ds_k \dots ds_1.
\end{aligned}$$

Since

$$\int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, s_1 v_1] \neq \emptyset\} t^{r-1} dt \mathcal{H}^{n-1}(du) = \frac{1}{r} s_1^r c(n, r),$$

the upper estimate follows easily.

For the lower estimate, we observe that $H(u, t) \cap [o, x_1, \dots, x_k] \neq \emptyset$ if and only if there is some $i \in \{1, \dots, k\}$ such that $H(u, t) \cap [o, x_i] \neq \emptyset$. Hence, $\mathbb{E}[V_n(Z_0)^k]$ can be estimated from below by

$$\begin{aligned}
& \int_{(\mathbb{R}^n)^k} \exp \left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \sum_{i=1}^k \mathbf{1}\{H(u, t) \cap [o, x_i] \neq \emptyset\} t^{r-1} dt \mathcal{H}^{n-1}(du) \right] dx_1 \dots dx_k \\
&= \left(\int_{\mathbb{R}^n} \exp \left[-\frac{2\gamma \|x_1\|^r}{n\kappa_n r} c(n, r) \right] dx_1 \right)^k.
\end{aligned}$$

Now the assertion can be shown by a straightforward calculation. \square

Remark 1. The lower bound in Proposition 1 can also be obtained by an application of Hölder's inequality. Moreover, for fixed r and k , the ratio of the upper and the lower bound is

$$\Gamma \left(\frac{kn}{r} + 1 \right) / \Gamma \left(\frac{n}{r} + 1 \right)^k = \text{const}(k, r) \cdot k^{\frac{kn}{r}}.$$

The following lemmas lead us successively to explicit formulas for the second moment and the variance of the volume of the zero cell. In a first step, the integral representation of the second moment of $V_n(Z_0)$ from (2) will be simplified considerably by an application of a Blaschke-Petkantschin formula (cf. [34]). For this purpose, we define

$$b_{n,2} := \frac{\omega_{n-1} \omega_n}{4\pi}.$$

For $1 \leq m \leq n$ and $x_1, \dots, x_m \in \mathbb{R}^n$, let $\nabla_m(x_1, \dots, x_m)$ denote the m -dimensional volume of the parallelepiped spanned by these vectors. Further, denote by ν_m^n the unique rotation invariant probability measure on $G(n, m)$.

Lemma 1. *We have*

$$\begin{aligned} \mathbb{E}[V_n(Z_0)^2] &= b_{n,2} \int_{\text{span}\{e_1, e_2\}^2} \exp \left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, x_1, x_2] \neq \emptyset\} t^{r-1} \right. \\ &\quad \left. \times dt \mathcal{H}^{n-1}(du) \right] \nabla_2(x_1, x_2)^{n-2} \lambda_{\text{span}\{e_1, e_2\}}^2(d(x_1, x_2)). \end{aligned}$$

Proof. From (2), the linear Blaschke-Petkantschin formula [34, Theorem 7.2.1], the rotation invariance of spherical Lebesgue measure and the invariance of $\nabla_2(\cdot, \cdot)$, we obtain that

$$\begin{aligned} &\mathbb{E}[V_n(Z_0)^2] \\ &= b_{n,2} \int_{G(n,2)} \int_{L^2} \exp \left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, x_1, x_2] \neq \emptyset\} \right. \\ &\quad \left. \times t^{r-1} dt \mathcal{H}^{n-1}(du) \right] \nabla_2(x_1, x_2)^{n-2} \lambda_L^2(d(x_1, x_2)) \nu_2^n(dL) \\ &= b_{n,2} \int_{SO_n} \int_{\text{span}\{e_1, e_2\}^2} \exp \left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, \vartheta x_1, \vartheta x_2] \neq \emptyset\} \right. \\ &\quad \left. \times t^{r-1} dt \mathcal{H}^{n-1}(du) \right] \nabla_2(\vartheta x_1, \vartheta x_2)^{n-2} \lambda_{\text{span}\{e_1, e_2\}}^2(d(x_1, x_2)) \nu_n(d\vartheta) \\ &= b_{n,2} \int_{SO_n} \int_{\text{span}\{e_1, e_2\}^2} \exp \left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(\vartheta u, t) \cap [o, \vartheta x_1, \vartheta x_2] \neq \emptyset\} \right. \\ &\quad \left. \times t^{r-1} dt \mathcal{H}^{n-1}(du) \right] \nabla_2(x_1, x_2)^{n-2} \lambda_{\text{span}\{e_1, e_2\}}^2(d(x_1, x_2)) \nu_n(d\vartheta) \\ &= b_{n,2} \int_{\text{span}\{e_1, e_2\}^2} \exp \left[-\frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, x_1, x_2] \neq \emptyset\} \right. \\ &\quad \left. \times t^{r-1} dt \mathcal{H}^{n-1}(du) \right] \nabla_2(x_1, x_2)^{n-2} \lambda_{\text{span}\{e_1, e_2\}}^2(d(x_1, x_2)), \end{aligned}$$

which yields the assertion of the lemma. \square

Next we simplify the inner double integral of the expression which was derived in Lemma 1 for $\mathbb{E}[V_n(Z_0)^2]$ by exploiting further the symmetry of the situation.

Lemma 2. *For $x_1, x_2 \in \text{span}\{e_1, e_2\} \subset \mathbb{R}^n$, we have*

$$\begin{aligned} &\int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, x_1, x_2] \neq \emptyset\} t^{r-1} dt \mathcal{H}^{n-1}(du) \\ &= \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{r+1}{2})} c(n, r) \int_0^{2\pi} \int_0^\infty \mathbf{1}\{H\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t\right) \cap [o, x_1, x_2] \neq \emptyset\} t^{r-1} dt d\theta. \end{aligned}$$

Proof. For $n = 2$ there is nothing to prove. Hence we can assume that $n \geq 3$. Let $E := \text{span}\{e_1, e_2\}$. The map

$$F : \begin{cases} S_E^1 \times (0, \frac{\pi}{2}) \times S_{E^\perp}^{n-3} & \rightarrow S^{n-1}, \\ (u_1, \theta, u_2) & \mapsto \cos(\theta)u_1 + \sin(\theta)u_2, \end{cases}$$

is injective and its image covers S^{n-1} up to a set of measure zero. Its Jacobian is

$$JF(u_1, \theta, u_2) = \cos(\theta)(\sin(\theta))^{n-3},$$

and hence the area-coarea formula (cf. [5, Theorem 3.2.22]) yields that

$$\begin{aligned} & \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, x_1, x_2] \neq \emptyset\} t^{r-1} dt \mathcal{H}^{n-1}(du) \\ &= \int_{S_E^1} \int_0^{\frac{\pi}{2}} \int_{S_{E^\perp}^{n-3}} \int_0^\infty \mathbf{1}\{H(\cos(\theta)u_1 + \sin(\theta)u_2, t) \cap [o, x_1, x_2] \neq \emptyset\} \\ & \quad \times \cos(\theta)(\sin \theta)^{n-3} t^{r-1} dt \mathcal{H}^{n-3}(du_2) d\theta \mathcal{H}^1(du_1). \end{aligned}$$

Since $[o, x_1, x_2] \subset E$ and

$$H(\cos(\theta)u_1 + \sin(\theta)u_2, t) \cap E = H_E\left(u_1, \frac{t}{\cos \theta}\right),$$

we get

$$\begin{aligned} & \int_{S^{n-1}} \int_0^\infty \mathbf{1}\{H(u, t) \cap [o, x_1, x_2] \neq \emptyset\} t^{r-1} dt \mathcal{H}^{n-1}(du) \\ &= (n-2)\kappa_{n-2} \int_{S_E^1} \int_0^{\frac{\pi}{2}} \int_0^\infty \mathbf{1}\{H_E\left(u_1, \frac{t}{\cos \theta}\right) \cap [o, x_1, x_2] \neq \emptyset\} \\ & \quad \times \cos(\theta)(\sin \theta)^{n-3} t^{r-1} dt d\theta \mathcal{H}^1(du_1) \\ &= (n-2)\kappa_{n-2} \int_{S_E^1} \int_0^{\frac{\pi}{2}} \int_0^\infty \mathbf{1}\{H_E(u_1, t) \cap [o, x_1, x_2] \neq \emptyset\} \\ & \quad \times (\cos \theta)^{r+1} (\sin \theta)^{n-3} t^{r-1} dt d\theta \mathcal{H}^1(du_1) \\ &= (n-2)\kappa_{n-2} \int_0^{\frac{\pi}{2}} (\cos \theta)^{r+1} (\sin \theta)^{n-3} d\theta \\ & \quad \times \int_{S^1} \int_0^\infty \mathbf{1}\{H(u_1, t) \cap [o, x_1, x_2] \neq \emptyset\} t^{r-1} dt \mathcal{H}^1(du_1) \\ &= \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} c(n, r) \int_0^{2\pi} \int_0^\infty \mathbf{1}\{H\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t\right) \cap [o, x_1, x_2] \neq \emptyset\} t^{r-1} dt d\theta, \end{aligned}$$

which completes the proof of the lemma. \square

Having simplified the inner integral of the expression found in Lemma 1 for $\mathbb{E}[V_n(Z_0)^2]$, we reduce the outer integral in the next lemma by again taking advantage of the problem's symmetry.

Lemma 3. *We have*

$$\begin{aligned}\mathbb{E}[V_n(Z_0)^2] &= 8\pi b_{n,2} \int_0^\pi \int_0^\infty \int_0^s \exp \left[-\frac{2\gamma c(n,r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\ &\quad \times \left. \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \right] \\ &\quad \times s^{n-1} u^{n-1} (\sin \varphi)^{n-2} du ds d\varphi.\end{aligned}$$

Proof. Combining Lemma 1 and Lemma 2, and introducing polar coordinates, we get

$$\begin{aligned}\mathbb{E}[V_n(Z_0)^2] &= b_{n,2} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^\infty \exp \left[-\frac{2\gamma c(n,r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\ &\quad \times \left. \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, u \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \right] \\ &\quad \times us |\sin(\varphi - \psi)|^{n-2} du ds d\varphi d\psi \\ &= b_{n,2} \int_0^{2\pi} \int_0^{2\pi} 2 \int_0^\infty \int_0^s \exp \left[-\frac{2\gamma c(n,r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\ &\quad \times \left. \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, u \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \right] \\ &\quad \times us |\sin(\varphi - \psi)|^{n-2} du ds d\varphi d\psi,\end{aligned}$$

where the symmetry in u and s is used to justify the second equality. Hence we derive

$$\begin{aligned}\mathbb{E}[V_n(Z_0)^2] &= 2b_{n,2} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^s \exp \left[-\frac{2\gamma c(n,r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\ &\quad \times \left. \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos(\theta - \psi) \\ \sin(\theta - \psi) \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos(\varphi - \psi) \\ \sin(\varphi - \psi) \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \right] \\ &\quad \times u^{n-1} s^{n-1} |\sin(\varphi - \psi)|^{n-2} du ds d\varphi d\psi \\ &= 4\pi b_{n,2} \int_0^{2\pi} \int_0^\infty \int_0^s \exp \left[-\frac{2\gamma c(n,r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\ &\quad \times \left. \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \right] \\ &\quad \times u^{n-1} s^{n-1} |\sin \varphi|^{n-2} du ds d\varphi \\ &= 8\pi b_{n,2} \int_0^\pi \int_0^\infty \int_0^s \exp \left[-\frac{2\gamma c(n,r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\ &\quad \times \left. \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \right] \\ &\quad \times u^{n-1} s^{n-1} |\sin \varphi|^{n-2} du ds d\varphi.\end{aligned}$$

$$\begin{aligned}
& \times \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \\
& \times s^{n-1} u^{n-1} (\sin \varphi)^{n-2} du ds d\varphi.
\end{aligned}$$

For the last equation, we used that

$$\begin{aligned}
& \int_{-\pi}^{2\pi} \int_0^\infty \int_0^s \exp \left[- \frac{2\gamma c(n, r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\
& \quad \times \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \\
& \quad \times u^{n-1} s^{n-1} |\sin \varphi|^{n-2} du ds d\varphi \\
& = \int_{-\pi}^{2\pi} \int_0^\infty \int_0^s \exp \left[- \frac{2\gamma c(n, r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\
& \quad \times \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos(-\varphi) \\ \sin(-\varphi) \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \\
& \quad \times u^{n-1} s^{n-1} |\sin(-\varphi)|^{n-2} (-1) du ds d\varphi \\
& = \int_0^\pi \int_0^\infty \int_0^s \exp \left[- \frac{2\gamma c(n, r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\
& \quad \times \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \\
& \quad \times u^{n-1} s^{n-1} (\sin \varphi)^{n-2} du ds d\varphi \\
& = \int_0^\pi \int_0^\infty \int_0^s \exp \left[- \frac{2\gamma c(n, r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\
& \quad \times \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \\
& \quad \times u^{n-1} s^{n-1} (\sin \varphi)^{n-2} du ds d\varphi \\
& = \int_0^\pi \int_0^\infty \int_0^s \exp \left[- \frac{2\gamma c(n, r)}{n\kappa_n} \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \right. \\
& \quad \times \int_0^{2\pi} \int_0^\infty \mathbf{1} \left\{ H \left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t \right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix} \right] \neq \emptyset \right\} t^{r-1} dt d\theta \\
& \quad \times u^{n-1} s^{n-1} (\sin \varphi)^{n-2} du ds d\varphi,
\end{aligned}$$

which completes the proof. \square

In the expression found in Lemma 3 the indicator function depends on θ, t, s, u and φ . Its

support can be determined explicitly. This is used in the proof of the following lemma where it is shown how the integration with respect to t in Lemma 3 can be carried out.

Lemma 4. For $u, s \in (0, \infty)$, $u \leq s$ and $\varphi \in (0, \pi)$,

$$\begin{aligned} & \int_0^{2\pi} \int_0^\infty \mathbf{1}\left\{H\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t\right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix}\right] \neq \emptyset\right\} t^{r-1} dt d\theta \\ &= \frac{1}{r} u^r \int_{-\frac{\pi}{2}}^{\alpha(\varphi, \frac{u}{s})} (\cos \theta)^r d\theta + \frac{1}{r} s^r \int_{\alpha(\varphi, \frac{u}{s}) - \varphi}^{\frac{\pi}{2}} (\cos \theta)^r d\theta, \end{aligned}$$

where

$$\alpha(\varphi, z) := \arctan\left(\frac{z - \cos \varphi}{\sin \varphi}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad (\varphi, z) \in (0, \pi) \times [0, 1].$$

Proof. We have to determine the support of the indicator function

$$\left\{ \begin{array}{ll} [0, 2\pi] \times [0, \infty) & \rightarrow \{0, 1\}, \\ (\theta, t) & \mapsto \mathbf{1}\left\{H\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t\right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix}\right] \neq \emptyset\right\}. \end{array} \right.$$

First, note that

$$H\left(\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, t\right) \cap \left[o, s \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix}\right] \neq \emptyset \quad (3)$$

is satisfied if and only if

$$t \in [0, u(\cos \theta)_+] \cup [0, s(\cos(\theta - \varphi))_+].$$

Hence we have to compare $(u \cos \theta)_+$ and $(s \cos(\theta - \varphi))_+$. We observe that

$$u \cos \theta = s \cos(\varphi - \theta) \Leftrightarrow \theta = \arctan\left(\frac{\frac{u}{s} - \cos \varphi}{\sin \varphi}\right) = \alpha\left(\varphi, \frac{u}{s}\right),$$

and therefore (3) is satisfied if and only if

$$(\theta, t) \in \left[-\frac{\pi}{2}, \alpha\left(\varphi, \frac{u}{s}\right)\right) \times [0, u \cos \theta] \cup \left(\left[\alpha\left(\varphi, \frac{u}{s}\right), \varphi + \frac{\pi}{2}\right] \times [0, s \cos(\theta - \varphi)]\right).$$

Now the integral can be easily computed. □

In the following remark, we collect some facts which are helpful for a proper understanding of the formulas for the second moment and the variance of the volume of the zero cell which are finally presented in Theorem 1.

Remark 2. (1) Let $\varphi \in (0, \pi)$ and $z \in [0, 1]$. Then

$$-\frac{\pi}{2} \leq \varphi - \frac{\pi}{2} = \alpha(\varphi, 0) \leq \alpha(\varphi, z) \leq \alpha(\varphi, 1) = \frac{\varphi}{2},$$

since $z \mapsto \alpha(\varphi, z)$ is increasing on $[0, 1]$.

(2) A simple calculation shows that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \theta)^r d\theta = \frac{\sqrt{\pi} \Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2} + 1)}.$$

(3) By the definition of κ_n and $b_{n,2}$, we have

$$n^2 \kappa_n^2 = 4\pi b_{n,2} \int_0^\pi (\sin \varphi)^{n-2} d\varphi.$$

In view of the next theorem and the consequences which will be derived from it, we provide a description of $(\mathbb{E}[V_n(Z_0)])^2$ as a double integral, i.e.

$$(\mathbb{E}[V_n(Z_0)])^2 = \frac{8\pi b_{n,2}}{r} \Gamma\left(\frac{2n}{r}\right) \left(\frac{n\kappa_n r}{2\gamma c(n, r)}\right)^{\frac{2n}{r}} \int_0^\pi \int_0^1 \frac{t^{n-1}}{(t^r + 1)^{\frac{2n}{r}}} (\sin \varphi)^{n-2} dt d\varphi. \quad (4)$$

This follows from the special case $k = 1$ of Proposition 1, Remark 2, (3), by the substitution $z = (t^r + 1)^{-1}$, and by basic calculations involving Beta and Gamma functions.

For the statement of the formulae for the second moment and the variance of $V_n(Z_0)$, we introduce the auxiliary function

$$F_r(t, \varphi) := \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{r+1}{2})} \left(t^r \int_{-\frac{\pi}{2}}^{\alpha(\varphi, t)} (\cos \theta)^r d\theta + \int_{\alpha(\varphi, t) - \varphi}^{\frac{\pi}{2}} (\cos \theta)^r d\theta \right)$$

for $(t, \varphi) \in [0, 1] \times (0, \pi)$. In the following, we will use that

$$\frac{1}{2} \leq F_r(t, \varphi) \leq t^r + 1 \leq 2, \quad (t, \varphi) \in [0, 1] \times (0, \pi), \quad (5)$$

which is implied by Remark 2, (1) and (2).

Theorem 1. *Let X be a Poisson hyperplane process in \mathbb{R}^n with an intensity measure of the form (1) with intensity $\gamma > 0$ and distance exponent $r \in (0, \infty)$. Then*

$$\mathbb{E}[V_n(Z_0)^2] = \frac{8\pi b_{n,2}}{r} \Gamma\left(\frac{2n}{r}\right) \left(\frac{n\kappa_n r}{2\gamma c(n, r)}\right)^{\frac{2n}{r}} \int_0^\pi \int_0^1 \frac{t^{n-1}}{F_r(t, \varphi)^{\frac{2n}{r}}} (\sin \varphi)^{n-2} dt d\varphi$$

and

$$\begin{aligned} \text{Var}[V_n(Z_0)] &= \frac{8\pi b_{n,2}}{r} \Gamma\left(\frac{2n}{r}\right) \left(\frac{n\kappa_n r}{2\gamma c(n, r)}\right)^{\frac{2n}{r}} \\ &\quad \times \int_0^\pi \int_0^1 \left(\frac{1}{F_r(t, \varphi)^{\frac{2n}{r}}} - \frac{1}{(t^r + 1)^{\frac{2n}{r}}} \right) t^{n-1} (\sin \varphi)^{n-2} dt d\varphi. \end{aligned}$$

Proof. The formula for $\mathbb{E}[V_n(Z_0)^2]$ is implied by Lemma 3, Lemma 4 and straightforward calculations. The formula for $\text{Var}[V_n(Z_0)^2]$ then follows from (4). \square

Remark 3. In Appendix A numerical calculations of the variance using the numerical integration functions of MATHEMATICA[®] are added. In small dimensions ($n = 2, 3, 4$) the numerical estimates of the variance for varying r are plotted for $\gamma = 1$. In this case observe that by Proposition 1 and Stirling's formula (see, e.g., [2]) we have $\lim_{r \rightarrow \infty} \mathbb{E}[V_n(Z_0)^k] = \kappa_n^k$ for $k \in \mathbb{N}$ and arbitrary fixed $n \geq 2$. In addition we study the choice $\gamma = \frac{n\kappa_n r}{2c(n, r)} \left(\Gamma(\frac{n}{r} + 1)\kappa_n\right)^{\frac{r}{n}}$, which implies $\mathbb{E}[V_n(Z_0)] = 1$ and $\lim_{r \rightarrow \infty} \mathbb{E}[V_n(Z_0)^k] = 1$ for $k \in \mathbb{N}$ and arbitrary fixed $n \geq 2$ by Proposition 1. Theorem 2, which will be proved in the following section, and Stirling's formula show that for both choices of γ we obtain $\text{Var}[V_n(Z_0)] = O(\frac{1}{r})$.

On the other hand for specific choices of r ($r = 1, r = 0.5n, r = n, r = 2n$) the numerical estimates of the variance for varying dimension n are plotted for $\gamma = 1$ and $\gamma = \frac{n\kappa_n r}{2c(n, r)} \left(\Gamma(\frac{n}{r} + 1)\kappa_n\right)^{\frac{r}{n}}$. The high-dimensional limiting behaviour of the moments and the variance is studied in the following section.

4 Variance estimates

In the next theorem, estimates for $\text{Var}[V_n(Z_0)]$ are provided. In these estimates two auxiliary quantities, $D(n, r)$ and $E(n, r)$, to be defined below, are involved. They are examined more closely in the following lemmas. For a better understanding of $E(n, r)$, we introduce the auxiliary function $M(v, r)$, $v \in [-\pi/2, \pi/2]$, for which an upper and a lower estimate is provided in Lemma 5. In Lemma 6 we establish estimates for $E(n, r)$ which are based on Lemma 5.

Definition 1. For $v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $r \in (0, \infty)$, we define

$$M(v, r) := \frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \int_v^{\frac{\pi}{2}} (\cos \theta)^r d\theta,$$

$$D(n, r) := \frac{n\kappa_n^2}{r} \Gamma\left(\frac{2n}{r} + 1\right) \left(\frac{n\kappa_n r}{4\gamma c(n, r)}\right)^{\frac{2n}{r}}$$

and

$$E(n, r) := \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^\pi (\sin \varphi)^{n-2} \int_0^1 \left[t^{n+r-1} \left(\frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \int_{\alpha(\varphi, t)}^{\frac{\pi}{2}} (\cos \theta)^r d\theta \right) \right. \\ \left. + t^{n-1} \left(\frac{\Gamma(\frac{r}{2} + 1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})} \int_{-\frac{\pi}{2}}^{\alpha(\varphi, t) - \varphi} (\cos \theta)^r d\theta \right) \right] dt d\varphi.$$

Since $0 \leq \varphi/2 \leq \varphi - \alpha(\varphi, t) \leq \pi/2$, if $\varphi \in (0, \pi)$, we can rewrite $E(n, r)$ in terms of the function $M(\cdot, r)$ as

$$E(n, r) = \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^\pi (\sin \varphi)^{n-2} \int_0^1 \left[t^{n+r-1} M(\alpha(\varphi, t), r) \right. \\ \left. + t^{n-1} M(\varphi - \alpha(\varphi, t), r) \right] dt d\varphi.$$

Theorem 2. *With these definitions, we have*

$$E(n, r) D(n, r) \leq \text{Var}[V_n(Z_0)] \leq E(n, r) D(n, r) 4^{\frac{2n}{r}+1}.$$

Proof. The mean value theorem and (5) yield that

$$\begin{aligned} & \int_0^\pi \int_0^1 \left(\frac{1}{F_r(t, \varphi)^{\frac{2n}{r}}} - \frac{1}{(t^r + 1)^{\frac{2n}{r}}} \right) t^{n-1} (\sin \varphi)^{n-2} dt d\varphi \\ & \geq \frac{n}{r} \frac{1}{2^{\frac{2n}{r}}} \int_0^\pi \int_0^1 \left(t^r + 1 - F_r(t, \varphi) \right) t^{n-1} (\sin \varphi)^{n-2} dt d\varphi. \end{aligned}$$

Similarly, again by the mean value theorem and (5) we have

$$\begin{aligned} & \int_0^\pi \int_0^1 \left(\frac{1}{F_r(t, \varphi)^{\frac{2n}{r}}} - \frac{1}{(t^r + 1)^{\frac{2n}{r}}} \right) t^{n-1} (\sin \varphi)^{n-2} dt d\varphi \\ & \leq 4 \frac{n}{r} 2^{\frac{2n}{r}} \int_0^\pi \int_0^1 \left(t^r + 1 - F_r(t, \varphi) \right) t^{n-1} (\sin \varphi)^{n-2} dt d\varphi. \end{aligned}$$

Now the assertion follows easily. □

In the next lemma, estimates for the function $M(\cdot, r)$ from Definition 1 are provided. Note that for fixed $v \in (0, \pi/2)$, the upper and the lower bound exhibit the same asymptotic behaviour as $r \rightarrow \infty$. Information about the asymptotic behaviour of $M(\cdot, r)$ is important for understanding the behaviour of the auxiliary quantity $E(n, r)$.

Lemma 5. *For $v \in [0, \frac{\pi}{2}]$ and $r \in (0, \infty)$, we have*

$$c \frac{1}{\sqrt{r}} (\cos v)^r \left(\frac{\pi}{2} - v \right) \leq M(v, r) \leq C \frac{1}{\sqrt{r}} \frac{(\cos v)^{r+1}}{\sin(v)}$$

for some constants $c, C > 0$ which are independent of v and r .

Proof. For the proof it is sufficient to consider the case $v \in (0, \frac{\pi}{2})$. As lower bound for the cosine in $(v, \frac{\pi}{2})$, we use the line through $\cos(v)$ at v and through 0 at $\frac{\pi}{2}$, which is the graph of the function

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad \theta \mapsto \frac{\cos(v)}{v - \frac{\pi}{2}} \left(\theta - \frac{\pi}{2} \right).$$

As upper bound for the cosine in $(v, \frac{\pi}{2})$, we use the tangent line at v , which is the graph of

$$g : \mathbb{R} \rightarrow \mathbb{R}, \quad \theta \mapsto -\sin(v) \left(\theta - v - \frac{\cos v}{\sin v} \right).$$

Now the assertion can be seen easily. □

The following lemma provides upper and lower bounds for the auxiliary quantity $E(n, r)$ from Definition 1 appearing in the estimates for $\text{Var}[V_n(Z_0)]$ in Theorem 2.

Lemma 6. (a) Let $r \in (0, \infty)$ be fixed. Then

$$c \leq E(n, r) \leq 2$$

with a constant $c > 0$ which depends on r but not on n .

(b) For all $r \in (0, \infty)$, we have

$$c \frac{1}{\sqrt{nr}} \left(\frac{2\sqrt{2}}{3} \right)^n \left(\frac{\sqrt{2}}{\sqrt{3}} \right)^r \leq E(n, r) \leq C \left(\frac{1}{\sqrt{r}} \left(\frac{2}{\sqrt{5}} \right)^r + \left(\frac{2}{\sqrt{5}} \right)^n \right)$$

with constants $c, C > 0$ which are independent of r and n .

Proof. (a) Let $r \in (0, \infty)$ be fixed. Remark 2, (2), implies that $0 \leq M(v, r) \leq 1$ for all $v \in [-\pi/2, \pi/2]$. Hence we get

$$E(n, r) \leq n \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^\pi (\sin \varphi)^{n-2} d\varphi \left(\frac{1}{n+r} + \frac{1}{n} \right) = \frac{n}{n+r} + 1 \leq 2.$$

On the other hand, we can estimate $E(n, r)$ from below by

$$\begin{aligned} E(n, r) &\geq n \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^{\frac{\pi}{2}} (\sin \varphi)^{n-2} \int_0^1 t^{n+r-1} M(\alpha(\varphi, t), r) dt d\varphi \\ &\geq n \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^{\frac{\pi}{2}} (\sin \varphi)^{n-2} d\varphi \cdot \frac{M(\frac{\pi}{4}, r)}{n+r} \\ &\geq (2(1+r))^{-1} M\left(\frac{\pi}{4}, r\right) =: c > 0. \end{aligned}$$

(b) We now consider the case of a parameter $r \in (0, \infty)$ which is not necessarily fixed and may depend on n . This case requires a more careful analysis. Our plan is to split the integrals with respect to φ and t and to make use of Lemma 5. A suitable choice of the parameters $t^* \in (0, 1)$ and $\varphi^* \in (0, \frac{\pi}{2})$, which determine where we split the integrals, is provided later. However, we should point out that our choice will be independent of r, n . First, we have the decomposition

$$\begin{aligned} E(n, r) &= \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^{\varphi^*} (\sin \varphi)^{n-2} \int_0^1 [t^{n+r-1} M(\alpha(\varphi, t), r) + t^{n-1} M(\varphi - \alpha(\varphi, t), r)] dt d\varphi \\ &\quad + \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{\varphi^*}^\pi (\sin \varphi)^{n-2} \left[\int_0^{t^*} \{t^{n+r-1} M(\alpha(\varphi, t), r) + t^{n-1} M(\varphi - \alpha(\varphi, t), r)\} dt \right. \\ &\quad \left. + \int_{t^*}^1 \{t^{n+r-1} M(\alpha(\varphi, t), r) + t^{n-1} M(\varphi - \alpha(\varphi, t), r)\} dt \right] d\varphi. \end{aligned}$$

For the subsequent estimate, we use Lemma 5, $0 \leq M(\cdot, \cdot) \leq 1$, the fact that $(\varphi, t) \mapsto \alpha(\varphi, t)$ is increasing in each variable separately and, in addition, that $\varphi - \alpha(\varphi, t) \geq \varphi^*/2$ for $\varphi \geq \varphi^*$, since $\alpha(\varphi, t) \leq \varphi/2$. Hence we get

$$\begin{aligned}
E(n, r) &\leq \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^{\varphi^*} (\sin \varphi)^{n-2} \left(\frac{1}{n+r} + \frac{1}{n} \right) d\varphi \\
&\quad + \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{\varphi^*}^{\pi} (\sin \varphi)^{n-2} \left(\frac{1}{n+r} (t^*)^{n+r} + \frac{1}{n} (t^*)^n \right) d\varphi \\
&\quad + \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{\varphi^*}^{\pi} (\sin \varphi)^{n-2} \left[\int_{t^*}^1 (t^{n+r-1} M(\alpha(\varphi^*, t^*), r) + t^{n-1} M(\frac{\varphi^*}{2}, r)) dt \right] d\varphi \\
&\leq \left(\frac{n}{n+r} + 1 \right) M\left(\frac{\pi}{2} - \varphi^*, n-2\right) + \frac{n}{n+r} ((t^*)^{n+r} + M(\alpha(\varphi^*, t^*), r)) \\
&\quad + (t^*)^n + M\left(\frac{\varphi^*}{2}, r\right) \\
&\leq C \frac{1}{\sqrt{r}} \max \{ \cos(\alpha(\varphi^*, t^*)), \cos\left(\frac{\varphi^*}{2}\right) \}^r + C \max \{ \sin(\varphi^*), t^* \}^n,
\end{aligned}$$

where we already used that our particular choice of t^*, φ^* will be independent of r, n . In fact, it seems to be difficult to choose t^* and φ^* in an optimal way as functions of r, n . However, in the special case $r = n$ the following particular choice turns out to be optimal. In any case, we fix

$$t^* := \frac{2}{\sqrt{5}}, \quad \varphi^* := \arcsin\left(\frac{2}{\sqrt{5}}\right).$$

For this choice we get

$$\sin(\varphi^*) = \frac{2}{\sqrt{5}} \approx 0.8944, \quad \cos\left(\frac{\varphi^*}{2}\right) \approx 0.8507, \quad \cos(\alpha(\varphi^*, t^*)) = \frac{2}{\sqrt{5}} \approx 0.8944.$$

Therefore our final result for the upper estimate of $E(n, r)$ is

$$E(n, r) \leq C \left(\frac{1}{\sqrt{r}} \left(\frac{2}{\sqrt{5}} \right)^r + \left(\frac{2}{\sqrt{5}} \right)^n \right)$$

with some constant $C > 0$, independent of r, n .

For the lower bound, we cut off the integral with respect to φ at $\varphi^\circ := 2 \arcsin(\frac{1}{\sqrt{3}})$. Then we have

$$\sin(\varphi^\circ) = \frac{2\sqrt{2}}{3} \approx 0.9428, \quad \cos\left(\frac{\varphi^\circ}{2}\right) = \frac{\sqrt{2}}{\sqrt{3}} \approx 0.8165.$$

It is not clear how to calculate the optimal choice of φ° for r depending on n in general. At least our choice is optimal for $\varphi^\circ \in (0, \frac{\pi}{2})$ and $r = n$. In any case, we thus obtain

$$\begin{aligned}
E(n, r) &\geq \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^{\varphi^\circ} (\sin \varphi)^{n-2} \int_0^1 [t^{n+r-1} M(\alpha(\varphi, t), r) + t^{n-1} M(\varphi - \alpha(\varphi, t), r)] dt d\varphi \\
&\geq n \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^{\varphi^\circ} (\sin \varphi)^{n-2} \int_0^1 t^{n+r-1} M\left(\frac{\varphi^\circ}{2}, r\right) dt d\varphi
\end{aligned}$$

$$\begin{aligned}
&= \frac{n}{n+r} M\left(\frac{\pi}{2} - \varphi^\circ, n-2\right) M\left(\frac{\varphi^\circ}{2}, r\right) \\
&\geq c \frac{1}{\sqrt{nr}} (\sin \varphi^\circ)^n (\cos \frac{\varphi^\circ}{2})^r,
\end{aligned}$$

where we used that for $0 \leq \varphi \leq \varphi^\circ$, we have $\alpha(\varphi, t) \leq \alpha(\varphi^\circ, t) \leq \frac{\varphi^\circ}{2}$, and therefore $M(\alpha(\varphi, t), r) \geq M(\frac{\varphi^\circ}{2}, r)$. Thus we finally arrive at

$$E(n, r) \geq c \frac{1}{\sqrt{nr}} \left(\frac{2\sqrt{2}}{3}\right)^n \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^r,$$

which concludes the proof of the lemma. \square

In the following corollary, the estimates in (6) are obtained from Proposition 1 by applying Stirling's formula. Similarly, the estimates in (7) are a consequence of Theorem 2, Lemma 6 and Stirling's formula.

Corollary 1. *For $k \in \mathbb{N}$, fixed $r \in (0, \infty)$ and some constants $c, C > 0$, depending on r but not on n , we have*

$$c \left(A(r) \frac{n}{\gamma} \left(1 + \frac{r}{n}\right)^{\frac{n}{2}} \right)^{\frac{kn}{r}} \leq \mathbb{E}[V_n(Z_0)^k] \leq C n^{\frac{1-k}{2}} \left(A(r) \frac{kn}{\gamma} \left(1 + \frac{r}{n}\right)^{\frac{n}{2}} \right)^{\frac{kn}{r}} \quad (6)$$

and

$$c \sqrt{n} \left(A(r) \frac{n}{\gamma} \left(1 + \frac{r}{n}\right)^{\frac{n}{2}} \right)^{\frac{2n}{r}} \leq \text{Var}[V_n(Z_0)] \leq C \sqrt{n} \left(A(r) \frac{4n}{\gamma} \left(1 + \frac{r}{n}\right)^{\frac{n}{2}} \right)^{\frac{2n}{r}}, \quad (7)$$

where

$$A(r) := \frac{\pi^{\frac{r+1}{2}}}{e \Gamma(\frac{r+1}{2})}.$$

For constant intensity γ , we infer from Corollary 1 that $\mathbb{E}[V_n(Z_0)^k]$ and $\text{Var}[V_n(Z_0)]$ are divergent for fixed r and $n \rightarrow \infty$. Hence, in the case of fixed distance exponent $r \in (0, \infty)$ and constant intensity, the moments $\mathbb{E}[V_n(Z_0)^k]$ are divergent. It is natural to ask if the intensity γ can be chosen as a function of n and r such that $\mathbb{E}[V_n(Z_0)]$ is equal to a positive constant λ^{-1} . By Proposition 1 we have

$$\mathbb{E}[V_n(Z_0)] = \Gamma\left(\frac{n}{r} + 1\right) \kappa_n \left(\frac{n \kappa_n r}{2\gamma c(n, r)} \right)^{\frac{n}{r}}.$$

Therefore, if we define

$$\widehat{\gamma}(r, n) := \frac{n \kappa_n r}{2c(n, r)} \left(\lambda \Gamma\left(\frac{n}{r} + 1\right) \kappa_n \right)^{\frac{r}{n}}$$

with $\lambda > 0$, then

$$\mathbb{E}[V_n(Z_0)] = \frac{1}{\lambda}.$$

If we plug $\widehat{\gamma}(r, n)$ into the lower estimate from Theorem 2, we deduce from Lemma 6 and Stirling's formula that (for fixed r)

$$\text{Var}[V_n(Z_0)] \geq c \cdot \frac{n}{\lambda^2} \frac{\Gamma\left(\frac{2n}{r} + 1\right)}{\Gamma\left(\frac{n}{r} + 1\right)^2} 2^{-\frac{2n}{r}} \geq c(r) \lambda^{-2} \sqrt{n} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

The preceding analysis suggests that in order to arrive at a limiting behaviour comparable to the case of a Poisson-Voronoi tessellation, i.e. with the variance converging to zero as $n \rightarrow \infty$, we cannot choose the distance parameter r fixed but have to adjust it to the dimension. In fact, in the following theorem we consider the case where r is proportional to n which is the natural choice in view of the estimates that have been obtained.

Corollary 2. *Let $r = an$ with constant $a \in (0, \infty)$ and $k \in \mathbb{N}$. Then there are constants $c, C > 0$, independent of n , such that*

$$c \frac{1}{\gamma^{\frac{k}{a}}} n^{\frac{k}{a} - \frac{k}{2}} \left(\frac{B(a)}{n} \right)^{\frac{kn}{2}} \leq \mathbb{E}[V_n(Z_0)^k] \leq C \frac{1}{\gamma^{\frac{k}{a}}} n^{\frac{k}{a} - \frac{k}{2}} \left(\frac{B(a)}{n} \right)^{\frac{kn}{2}} \quad (8)$$

and

$$\begin{aligned} c \frac{1}{\sqrt{an}} \left(\frac{2\sqrt{2}}{3} \right)^n \left(\frac{\sqrt{2}}{\sqrt{3}} \right)^{an} \frac{1}{\gamma^{\frac{2}{a}}} n^{\frac{2}{a} - 1} \left(\frac{B(a)}{n} \right)^n \\ \leq \text{Var}[V_n(Z_0)] \\ \leq C \left(\frac{1}{\sqrt{an}} \left(\frac{2}{\sqrt{5}} \right)^{an} + \left(\frac{2}{\sqrt{5}} \right)^n \right) \frac{1}{\gamma^{\frac{2}{a}}} n^{\frac{2}{a} - 1} \left(\frac{B(a)}{n} \right)^n, \end{aligned} \quad (9)$$

where

$$B(a) := \frac{2\pi e(a+1)^{\frac{a+1}{a}}}{a}.$$

Hence, if the intensity γ is constant and $r = an$, then $\mathbb{E}[V_n(Z_0)^k]$ and $\text{Var}[V_n(Z_0)]$ are converging to zero as $n \rightarrow \infty$.

Proof. The inequalities in (8) follow from Proposition 1 by means of Stirling's formula.

The inequalities in (9) follow from Theorem 2, by estimating $D(n, an)$ by means of Stirling's formula, and by estimating $E(n, an)$ by means of Lemma 6 (b). \square

Now we choose again $r = an$, with fixed $a > 0$, and determine the intensity γ in such a way that the expected volume of the zero cell is equal to a positive constant. This is possible for a special intensity function depending on a , the dimension n and a positive constant λ that can be prescribed arbitrarily. In the following theorem, we describe the asymptotic behaviour of the volume of the zero cell in such a setting.

Theorem 3. *Let $r = an$, with constant $a \in (0, \infty)$, and let the intensity be chosen as*

$$\hat{\gamma}(a, n) = \frac{an^2 \kappa_n}{2c(n, an)} \left(\lambda \Gamma \left(\frac{1}{a} + 1 \right) \kappa_n \right)^a$$

with $\lambda > 0$. Then the following is true.

(a) For $k \in \mathbb{N}$,

$$\frac{1}{\lambda^k} \leq \mathbb{E}[V_n(Z_0)^k] \leq \frac{\Gamma(\frac{k}{a} + 1)}{\lambda^k \Gamma(\frac{1}{a} + 1)^k};$$

in particular,

$$\mathbb{E}[V_n(Z_0)] = \frac{1}{\lambda}.$$

(b) *There are constants $c, C > 0$, independent of n , such that*

$$c \frac{1}{n} \left(\frac{2\sqrt{2}}{3} \right)^n \left(\frac{\sqrt{2}}{\sqrt{3}} \right)^{an} \leq \text{Var}[V_n(Z_0)] \leq C \left(\frac{1}{\sqrt{n}} \left(\frac{2}{\sqrt{5}} \right)^{an} + \left(\frac{2}{\sqrt{5}} \right)^n \right);$$

in particular,

$$\lim_{n \rightarrow \infty} \text{Var}[V_n(Z_0)] = 0.$$

Proof. The inequalities in (a) follow from Lemma 1 and the special choice of the intensity as $\hat{\gamma}(a, n)$. The inequalities in (b) follow from Theorem 2, since $D(n, an)$ is constant for the intensity chosen as $\hat{\gamma}(a, n)$ and by estimating $E(n, an)$ by means of Lemma 6 (b). \square

A Numerical Calculations

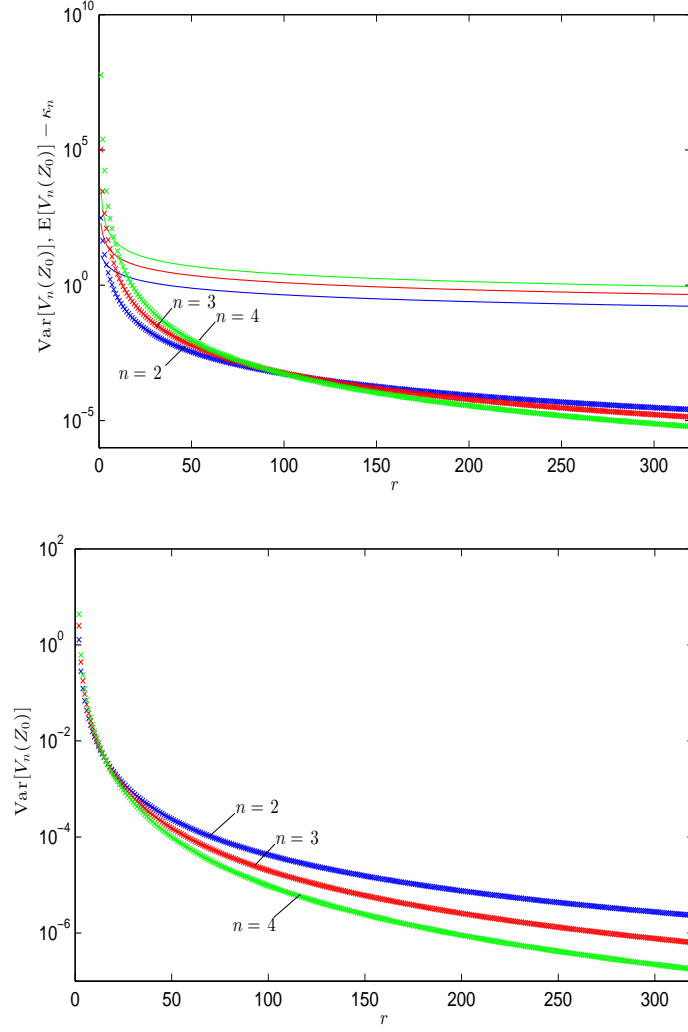


Figure 1: Numerical evaluation of the formula for the variance from Theorem 1 using the numerical integration functions of MATHEMATICA[®]. In the top panel, we fix $\gamma = 1$ and $\mathbb{E}[V_n(Z_0)] - \lim_{r \rightarrow \infty} \mathbb{E}[V_n(Z_0)] = \mathbb{E}[V_n(Z_0)] - \kappa_n$ is plotted as a solid line in the corresponding color for comparison. In the bottom panel, γ is chosen in such a way that $\mathbb{E}[V_n(Z_0)] = 1$.

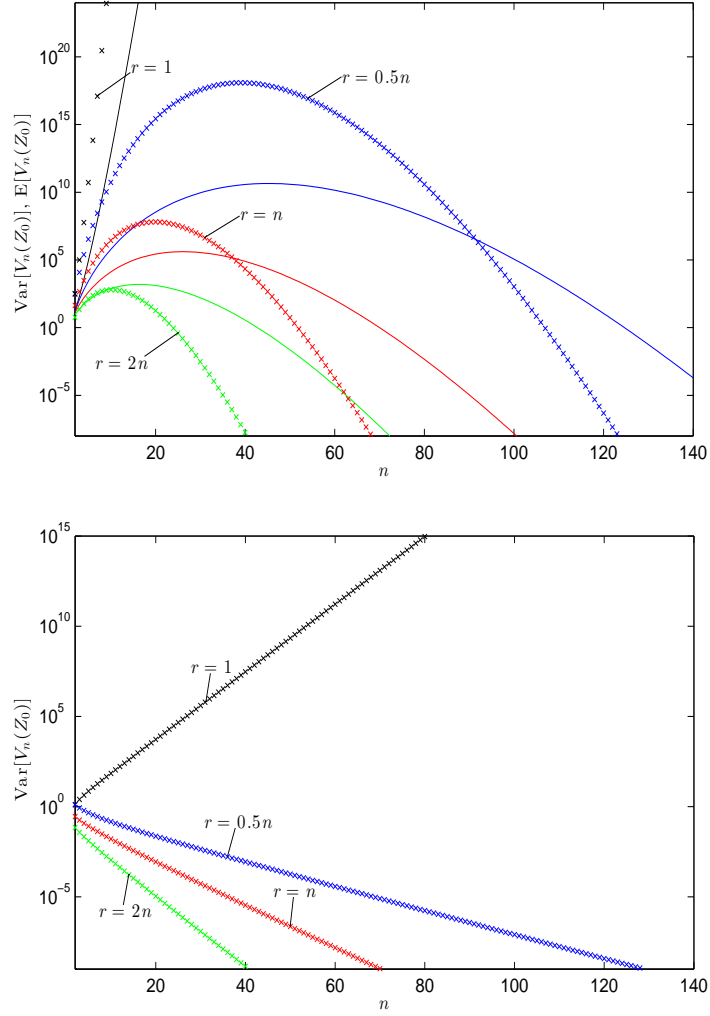


Figure 2: Numerical evaluation of the formula for the variance from Theorem 1 using the numerical integration functions of MATHEMATICA[®]. In the top panel, we fix $\gamma = 1$ and $\mathbb{E}[V_n(Z_0)]$ is plotted as a solid line in the corresponding color for comparison. In the bottom panel, γ is chosen in such a way that $\mathbb{E}[V_n(Z_0)] = 1$.

Acknowledgement

The authors would like to thank Walter Mickel (Department of Mathematics, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany) for his valuable advice concerning the numerical calculations.

References

- [1] ALISHAHI, K. AND SHARIFITABAR, M. (2008). Volume degeneracy of the typical cell and the chord length distribution for Poisson-Voronoi tessellations in high dimensions. *Adv. Appl. Prob.* **40**, 919–938.
- [2] ARTIN, E. (1964). *The Gamma Function*. Holt, Rinehart and Winston, New York.
- [3] BÁRÁNY, I. (2008). Random points and lattice points in convex bodies. *Bull. Amer. Math. Soc.* **45**, 339–365.

- [4] CALKA, P. (2010). Tessellations. In *New Perspectives in Stochastic Geometry*. ed. W. S. Kendall and I. Molchanov. Oxford Univ. Press., Oxford, pp. 145–169.
- [5] FEDERER, H. (1969). *Geometric Measure Theory*. Springer, Berlin.
- [6] HILHORST, H.J. AND CALKA, P. (2008). Random line tessellations of the plane: statistical properties of many-sided cells. *J. Stat. Phys.* **132**, 627–647.
- [7] HUG, D. (2007). Random Mosaics. In *Stochastic Geometry*. ed. W. Weil. Springer, Berlin pp. 247–266.
- [8] HUG, D., REITZNER, M. AND SCHNEIDER, R. (2004). Large Poisson-Voronoi cells and Crofton cells. *Adv. Appl. Prob.* **36**, 667–690.
- [9] HUG, D., REITZNER, M. AND SCHNEIDER, R. (2004). The limit shape of the zero cell in a stationary Poisson hyperplane tessellation. *Ann. Prob.* **32**, 1140–1167.
- [10] HUG, D. AND SCHNEIDER, R. (2007). Asymptotic shapes of large cells in random tessellations. *Geom. Funct. Anal.* **17**, 156–191.
- [11] HUG, D. AND SCHNEIDER, R. (2007). Typical cells in Poisson hyperplane tessellations. *Discrete Comput. Geom.* **38**, 305–319.
- [12] HUG, D. AND SCHNEIDER, R. (2010). Large faces in Poisson hyperplane mosaics. *Ann. Prob.* **38**, 1320–1344.
- [13] MATHAI, A. M. (1999). *An Introduction to Geometrical Probability*. Gordon and Breach, Amsterdam.
- [14] MECKE, J. (1986). On some inequalities for Poisson networks. *Math. Nachr.* **128**, 81–86.
- [15] MECKE, J. (1988). Random r -flats meeting a ball. *Arch. Math.* **51**, 378–384.
- [16] MECKE, J. (1991). On the intersection density of flat processes. *Math. Nachr.* **151**, 69–74.
- [17] MECKE, J. (1995). Inequalities for the anisotropic Poisson polytope. *Adv. Appl. Prob.* **27**, 56–62.
- [18] MECKE, J. (1998). Inequalities for mixed stationary Poisson hyperplane tessellations. *Adv. Appl. Prob.* **30**, 921–928.
- [19] MECKE, J. (1999). On the relationship between the 0-cell and the typical cell of a stationary random tessellation. *Pattern Recogn.* **32**, 1645–1648.
- [20] MEIJERING, J. L. (1953). Interface area, edge length, and number of vertices in crystal aggregates with random nucleation. *Philips Res. Rep.* **8**, 270–290.
- [21] MILES, R. E. (1974). A synopsis of ‘Poisson flats in euclidean spaces’. *Stochastic Geometry* 202–227.
- [22] MILES, R. E. (1984). Sectional Voronoi tessellations. *Revista de la Unión Matemática Argentina* **29**, 310–327.
- [23] MILES, R. E. AND MAILLARD, R. J. (1982). The basic structures of Voronoi and generalized Voronoi polygons. *J. Appl. Prob.* **19A**, 97–111.
- [24] MØLLER, J. (1989). Random tessellations in \mathbb{R}^d . *Adv. Appl. Prob.* **21**, 37–73.
- [25] MØLLER, J. (1994). *Lectures on Random Voronoi Tessellations*. vol. 87 of *Lecture Notes in Statistics*. Springer, New York.
- [26] MUCHE, L. (2010). Contact and chord length distribution functions of the Poisson-Voronoi tessellation in high dimensions. *Adv. Appl. Prob.* **42**, 48–68.
- [27] MUCHE, L. AND BALLANI, F. (2010). The second volume moment of the typical cell and higher moments of edge lengths of the spatial Poisson-Voronoi tessellation. *Monatsh. Math.* **163**, 71–80.
- [28] NEWMAN, C.M., RINOTT, Y. AND TVERSKY, A. (1983). Nearest neighbors and Voronoi regions in certain point processes. *Adv. Appl. Prob.* **15**, 726–751.

- [29] NEWMAN, C.M. AND RINOTT, Y. (1985). Nearest neighbors and Voronoi volumes in high-dimensional point processes with various distance functions. *Adv. Appl. Prob.* **17**, 794–809.
- [30] OKABE, A., BOOTS, B., SUGIHARA, K. AND CHIU, S. (2000). *Spatial tessellations: Concepts and Applications of Voronoi Diagrams*. Wiley & Sons, Chichester.
- [31] REITZNER, M. (2010). Random Polytopes. In *New Perspectives in Stochastic Geometry*. ed. W. S. Kendall and I. Molchanov. Oxford Univ. Press., Oxford, pp. 45–76.
- [32] SCHNEIDER, R. (2003). Nonstationary Poisson hyperplanes and their induced tessellations. *Adv. Appl. Prob.* **35**, 139–158.
- [33] SCHNEIDER, R. (2008). Recent results on random polytopes. *Boll. Un. Math. Ital.* **1**, 17–39.
- [34] SCHNEIDER, R. AND WEIL, W. (2008). *Stochastic and Integral Geometry*. Springer, Berlin.
- [35] SPODAREV, E. (2001). On the rose of intersections of stationary flat processes. *Adv. Appl. Prob.* **33**, 584–599.
- [36] STOYAN, D., KENDALL, W. S. AND MECKE, J. (1995). *Stochastic Geometry and its Applications*. 2nd ed. Wiley, Chichester.
- [37] YAO, Y.-C. (2010). On variances of partial volumes of the typical cell of a Poisson-Voronoi tessellation and large-dimensional volume degeneracy. *Adv. Appl. Prob.* **42**, 359–370.